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Exponential Codimension Growth of PI Algebras: An Exact Estimate

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Let A be an associative PI-algebra over a field F of characteristic zero. By studying the exponential behavior of the sequence of codimensions $\{c_n(A)\}$ of A, we prove that $Inv(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$ always exists and is an integer. We also give an explicit way for computing such integer: let B be a finite dimensional $\mathbf{Z}_2$-graded algebra whose Grassmann envelope $G(B)$ satisfies the same identities of A; then $Inv(A) = Inv(G(B)) = \dim C^{(0)} + \dim C^{(1)}$ where $C^{(0)} + C^{(1)}$ is a suitable $\mathbf{Z}_2$-graded semisimple subalgebra of B. © 1999 Academic Press

1. INTRODUCTION

Let A be an associative algebra over a field F of characteristic zero satisfying a polynomial identity (PI-algebra). If $F\langle X\rangle = F\langle x_1, x_2, \ldots\rangle$ is the free associative algebra of countable rank, the identities of A form a T-ideal $Id(A)$ of $F\langle X\rangle$ and $F\langle X\rangle/Id(A)$ is the relatively free algebra of countable rank in the variety $\mathcal{V}(A)$ generated by A. Since char $F=0$, $Id(A)$ (and $F\langle X\rangle/Id(A)$) is determined by its multilinear polynomials. Let V_n be the space of multilinear polynomials in $x_1, \ldots, x_n$; there is a natural action of the symmetric group S_n on $V_n/V_n \cap Id(A)$ and the representation theory of S_n is exploited in order to study this space. Let $c_n(A) = \dim_F V_n/V_n \cap Id(A)$ be the nth codimension of A. The basic property of the sequence

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$\{c_n(A)\}_{n\geqslant 1}$ is that it is exponentially bounded: Regev in [8] proved that if A is a PI-algebra, then there exist a, $\alpha>0$ such that $c_n(A)\leqslant a\alpha^n$ for all n. One also defines a notion of growth of the variety $\mathscr{V}(A)$ as the growth of the sequence $\{c_n(A)\}$.

The description of the sequence of codimensions in general seems to be a very difficult problem at present. Its asymptotic behavior has been computed for some significant classes of algebras (see [3, 9]) and it has been conjectured that $c_n(A) \underset{n\to\infty}{\simeq} an^b\alpha^n$ for some constants a, b, α ([10]).

To capture the exponential growth of $c_n(A)$ we make the following

DEFINITION. For any PI-algebra A define

$$\overline{Inv(A)}=\limsup_{n\to\infty} \sqrt[n]{c_n(A)}, \qquad \underline{Inv(A)}=\liminf_{n\to\infty} \sqrt[n]{c_n(A)}$$

and, in case of equality,

$$Inv(A)=\overline{Inv(A)}=\underline{Inv(A)}.$$

A conjecture, well known to mathematicians working in PI-theory, states that for any PI-algebra A, $Inv(A)$ exists and is an integer. The main result of this paper confirms this conjecture; in fact we prove

THEOREM 1. *Let A be a PI-algebra over a field of characteristic zero. Then $Inv(A)$ exists and is an integer.*

As an immediate corollary we discover again the precise value of $Inv(A)$ for A a verbally prime PI-algebra ([3, 9]).

While proving Theorem 1 we actually give an explicit way of computing $Inv(A)$; more precisely by a well known theorem of Kemer ([7]) given any PI-algebra A, there exists a finite dimensional $\mathbf{Z}_2$-graded algebra B whose Grassmann envelope $G(B)$ satisfies the same identities of A. Then $Inv(A)=Inv(G(B))=\dim C^{(0)}+\dim C^{(1)}$ where $C^{(0)}+C^{(1)}$ is a suitable $\mathbf{Z}_2$-graded semisimple subalgebra of B.

We should remark that the above theorem has been recently proved for a finitely generated PI-algebra in [5]. Also in that paper it was proved that if A is a finite dimensional algebra, then A is central simple over F if and only if $Inv(A)=\dim_F A$.

2. PRELIMINARIES

Throughout F will be a field of characteristic zero, $F\langle X\rangle$ the free associative algebra over F with infinite set $X=\{x_1, x_2, ...\}$ of free generators. Sometimes we denote for convenience some free generators by other letters x_i^j, y_i, ...

We let S_n be the symmetric group on $\{1, ..., n\}$ and $V_n = V_n(x_1, ..., x_n) = \mathrm{Span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$ the space of multilinear polynomials in $x_1, ..., x_n$. For $\sigma \in S_n$ the map $\sigma \to x_{\sigma(1)} \cdots x_{\sigma(n)}$ induces a linear isomorphism $FS_n \cong V_n$ and this in turn defines a left action of S_n on V_n (usually denoted by $\circ$): if $\sigma \in S_n$, $f(x_1, ..., x_n) \in V_n$, $\sigma \circ f(x_1, ..., x_n) = f(x_{\sigma(1)}, ..., x_{\sigma(n)})$. Also, if $Id(A)$ is the T-ideal of identities of an algebra A, $V_n \cap Id(A)$ is invariant under this action and we regard $V_n / V_n \cap Id(A)$ as a left S_n-module. Let $\chi_n(A)$ be its S_n-character and $c_n(A) = \dim(V_n / V_n \cap Id(A))$ the corresponding degree. $\chi_n(A)$ and $c_n(A)$ are called the *nth cocharacter* and the *nth codimension* of A respectively.

Recall that the representation theory of S_n is related to the theory of partitions of n. Let $\lambda \vdash n$ be a partition of n and χ_λ the irreducible character associated to λ. Write $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ where $m_\lambda \geqslant 0$ is the multiplicity of χ_λ in $\chi_n(A)$. By taking degrees we get $c_n(A) = \sum_{\lambda \vdash n} m_\lambda d_\lambda$ and $d_\lambda = \chi_\lambda(1)$ can be computed either by the Young–Frobenius formula or by the hook formula.

The technique for proving that $Inv(A) = p$ will be that of finding constants a_1, a_2, b_1, b_2 such that

$$a_1 n^{b_1} p^n \leqslant c_n(A) \leqslant a_2 n^{b_2} p^n.$$

Both upper and lower bound are computed by studying the decomposition $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$. The upper bound will follow from two basic facts: (1) the multiplicities m_λ are polynomially bounded ([1]); (2) if χ_λ appears in $\chi_n(A)$ with non-zero multiplicity, then the associated Young diagram is constrained in an infinite hook of fixed height and width. The main ingredients for computing the lower bound are the central polynomials for $n \times n$ matrices constructed in [4] and the Littlewood–Richardson rule.

In order to compute $Inv(A)$, a basic reduction is provided by a fundamental theorem of Kemer that we now describe. Let G be the Grassmann algebra on a countable dimensional vector space over F. G has a natural $\mathbf{Z}_2$-grading, $G = G^{(0)} + G^{(1)}$ where $G^{(0)}$ and $G^{(1)}$ are the spaces generated by the even degree and odd degree monomials respectively. If $A = A^{(0)} + A^{(1)}$ is a $\mathbf{Z}_2$-graded algebra over F, then $G(A) = A^{(0)} \otimes G^{(0)} + A^{(1)} \otimes G^{(1)}$ is called *the Grassmann envelope* of A. The following theorem holds

Theorem 2 (Kemer [7, Theorem 2.3]). *If A is any PI-algebra then there exists a finite dimensional $\mathbf{Z}_2$-graded algebra B such that $Id(A) = Id(G(B))$.*

3. S_n-REPRESENTATIONS AND HOOKS

Let $\lambda \vdash n$; we will usually identify λ with the corresponding Young diagram D_λ. If $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k) \vdash n$ and $\lambda' = (\lambda'_1, \lambda'_2, ..., \lambda'_t) \vdash n'$ then we say that $\lambda \geqslant \lambda'$ if $k \geqslant t$ and $\lambda_1 \geqslant \lambda'_1$, $\lambda_2 \geqslant \lambda'_2, ...$ (i.e., D_λ contains D'_λ as a subdiagram in its upper left corner).

Recall that if $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k) \vdash n$, then $h(\lambda) = k$ is the height of λ and $l(\lambda) = \lambda_1$ is the length of λ.

Let T_λ be a Young tableau and

$$e_{T_\lambda} = \sum_{\substack{\sigma \in R_{T_\lambda} \\ \tau \in C_{T_\lambda}}} (-1)^\tau \sigma\tau = \left(\sum_{\sigma \in R_{T_\lambda}} \sigma \right) \left(\sum_{\tau \in C_{T_\lambda}} (-1)^\tau \tau \right)$$

the corresponding essential idempotent of FS_n, where R_{T_λ} is the subgroup of S_n of row permutations of T_λ and C_{T_λ} is the subgroup of column permutations of T_λ.

In the next lemmas we record two facts needed in the sequel.

Lemma 1. *Let H be a subgroup of C_{T_λ}, M an FS_n-module and $e_{T_\lambda} u \neq 0$ for some $u \in M$. Then*

$$\left(\sum_{\sigma \in H} (-1)^\sigma \sigma \right) e_{T_\lambda} u \neq 0.$$

Proof. Write $C_{T_\lambda} = a_1 H \cup a_2 H \cup \cdots \cup a_m H$ where $a_1 = 1, a_2, ..., a_m$ is a left transversal of H in C_{T_λ} and let $r = \sum_{\sigma \in H} (-1)^\sigma \sigma$. If $r e_{T_\lambda} u = 0$ then $a_i r e_{T_\lambda} u = 0$ in M for all $i = 1, ..., m$. Hence

$$\begin{aligned} e_{T_\lambda}^2 u &= \left(\sum_{\tau \in R_{T_\lambda}} \tau \right) \left(\sum_{\sigma \in C_{T_\lambda}} (-1)^\sigma \sigma \right) e_{T_\lambda} u \\ &= \left(\sum_{\tau \in R_{T_\lambda}} \tau \right) (a_1 r e_{T_\lambda} u \pm \cdots \pm a_m r e_{T_\lambda} u) = 0, \end{aligned}$$

a contradiction since $e_{T_\lambda}^2 = \gamma e_{T_\lambda}$ for some integer $\gamma \neq 0$. ∎

A slight modification of the previous argument gives the proof of the following

Lemma 2. *Let H be a subgroup of R_{T_λ}, M an FS_n-module and $e_{T_\lambda} u \neq 0$ for some $u \in M$. Then*

$$\left(\sum_{\sigma \in H} \sigma \right) \left(\sum_{\tau \in C_{T_\lambda}} (-1)^\tau \tau \right) e_{T_\lambda} u \neq 0.$$

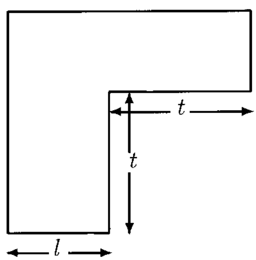
Let $\lambda \vdash n$ and T_λ a tableau associated to λ. We make the following

DEFINITION. A multilinear polynomial $f = f(x_1, ..., x_n)$ *corresponds* to T_λ if $f = e_{T_\lambda} \circ f_0$ for some multilinear polynomial $f_0 \in V_n$.

Given integers l, d, $t \geqslant 0$ we define

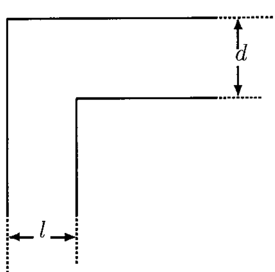
$$h(l, d, t) = (\underbrace{l+t, ..., l+t}_{d}, \underbrace{l, ..., l}_{t}).$$

It is clear that the diagram associated to $h(l, d, t)$ is hook shaped (see picture)



We also define an infinite hook $H(l, d)$ as follows

$$H(l, d) = \bigcup_{t \geqslant 1} h(l, d, t)$$



The following two lemmas record important properties of the hooks.

LEMMA 3. *Let $\lambda \vdash n$ be such that $\lambda \geqslant h(l, d, t)$ for some l, d, t. If $f(x_1, ..., x_n)$ is a multilinear polynomial corresponding to a tableau T_λ, then there exist $r \in FS_n$ such that $r \circ f \neq 0$ and a subset Y of $\{x_1, ..., x_n\}$ such that*

1. *$Y = Y_1 \cup \cdots \cup Y_d$, $r \circ f$ is symmetric in each set of variables Y_i, $i = 1, ..., d$, and $|Y_i| = t + l$;*

2. *$r \circ f$ can be decomposed into a sum of multilinear polynomials $r \circ f = f_1 + f_2 + \cdots + f_k$ such that for every f_i there is a partition $Y = Y'_1 \cup \cdots \cup Y'_{t+l}$ with $|Y'_j| = d$ and f_i is alternating in each set of variables Y'_j, $j = 1, ..., t + d$.*

Proof. By hypothesis the tableau T_λ contains a rectangular tableau T_0 with d rows and $t + l$ columns. Let N_j, $j = 1, ..., d$ be the set of integers in the jth row of T_0, N'_i, $i = 1, ..., t + l$, the set of integers in the ith column of T_0 and $N = N_1 \cup \cdots \cup N_d$. Set $H = \{\sigma \in R_{T_\lambda} \mid \sigma(i) = i \text{ for any } i \notin N\}$ and

$$r_0 = \sum_{\tau \in C_{T_\lambda}} (-1)^\tau \tau,$$

$$r = \left(\sum_{\sigma \in H} \sigma \right) r_0.$$

Let $Y_j = \{x_i \mid i \in N_j\}$, $Z_i = \{x_s \mid s \in N'_i\}$.

Clearly the element $g = r \circ f$ is symmetric in the variables from Y_j for each j. On the other hand, the polynomial $r_0 \circ f$ is alternating in the variables of each Z_i, therefore for any $\sigma \in H$ the element $\sigma \circ r_0 \circ f$ is alternating in the variables of each $Y'_i = \sigma(Z_i)$ for every $i = 1, ..., t + l$. Hence, $r \circ f$ is the required multilinear polynomial. By Lemma 2, $r \circ f$ is non-zero and the proof is complete. ∎

We remark that in the previous proof we can choose N_j and N'_i to be any set of integers in the jth row and in the ith column of T_0 respectively.

LEMMA 4. *Let $f(x_1, ..., x_n)$ correspond to the Young tableau T_λ and suppose that $\lambda \geqslant h(l, d, t)$. Then for some $r \in FS_n$, $r \circ f \neq 0$ and there is a subset Y of $\{x_1, ..., x_n\}$ such that*

1. *$Y = Y_1 \cup \cdots \cup Y_l$, $r \circ f$ is alternating in each set of variables Y_i, $i = 1, ..., l$, and $|Y_i| = t + d$;*

2. *$r \circ f$ can be decomposed into a sum of multilinear polynomials $r \circ f = f_1 + f_2 + \cdots + f_k$ such that for every f_i there is a partition $Y = Y'_1 \cup \cdots \cup Y'_{t+d}$ with $|Y'_j| = l$ and f_i is symmetric in each set of variables Y'_j, $j = 1, ..., t + l$.*

Proof. As in the previous Lemma T_λ contains a rectangular tableau T_0 with $d+t$ rows and l columns. Let N_j, $j=1, ..., l$ be the set of integers in the jth column of T_0, N'_i, $i=1, ..., t+d$, the set of integers in the ith row of T_0 and $N=N_1 \cup \cdots \cup N_l$. Denote $H=\{\sigma \in C_{T_\lambda} \mid \sigma(i)=i \text{ for any } i \notin N\}$ and

$$r=\left(\sum_{\sigma \in H} (-1)^\sigma \sigma\right).$$

Let $Y_j=\{x_i \mid i \in N_j\}$, $Z_i=\{x_s \mid s \in N'_i\}$.

By definition the element f is symmetric in the variables from Z_i. Hence, for any $\sigma \in H$ the polynomial $\sigma \circ f$ is symmetric in the variables from $Y'_i=\sigma(Z_i)$. This implies the second part of the lemma. Also, the polynomial $r \circ f$ is non-zero by Lemma 1 and is alternating on the variables from Y_j for any $j=1, ..., l$. ∎

Recall that for a partition $\lambda \vdash n$, $d_\lambda=\chi_\lambda(1)$ is the degree of the irreducible S_n-character associated to λ. Next result easily follows from the hook formula for d_λ.

LEMMA 5. *Let $\lambda \vdash n$, $\lambda' \vdash n'$ be such that $\lambda \leqslant \lambda'$. If $n-n' \leqslant c$ then $n^{-2c} d_{\lambda'} \leqslant d_\lambda \leqslant n^c d_{\lambda'}$.*

Remark. Using the structure of the essential idempotent e_{T_λ} one can find that in the previous lemma $d_{\lambda'} \leqslant d_\lambda$.

LEMMA 6 [2]. *For some constants C, $r>0$ the following inequality holds*

$$\sum_{\substack{\lambda \vdash n \\ \lambda \in H(l,\, d)}} d_\lambda \leqslant C n^r (l+d)^n.$$

4. COMPUTING THE UPPER BOUND

Throughout this section F will be an algebraically closed field of characteristic zero and $A=A^{(0)}+A^{(1)}$ a finite dimensional $\mathbf{Z}_2$-graded algebra over F. Let $A=B+J$ be the Wedderburn decomposition of A where B is a semisimple subalgebra of A and $J=J(A)$ its Jacobson radical. It is well known (see [7, p. 21]) that J is a graded ideal and $B=A_1 \oplus \cdots \oplus A_K$ where $A_1, ..., A_K$ are homogeneous in $\mathbf{Z}_2$-grading simple subalgebras and for each $i=1, ..., K$, $A_i=A_i^{(0)}+A_i^{(1)}$ is the induced $\mathbf{Z}_2$-grading. From now on we let $m=\dim A$ and $J^q=0$.

Consider all possible products of the type

$$B_1 J B_2 J \cdots J B_k \neq 0 \tag{1}$$

where $B_1, ..., B_k$ are distinct $A_1, ..., A_K$ and define

$$p^{(0)} = \dim(B_1^{(0)} \oplus \cdots \oplus B_k^{(0)}),$$

$$p^{(1)} = \dim(B_1^{(1)} \oplus \cdots \oplus B_k^{(1)}).$$

Now denote by p the maximal value of $p^{(0)} + p^{(1)}$ where $B_1, ..., B_k$ satisfy (1).

LEMMA 7. *Let $B_1, ..., B_t$ be not necessarily distinct subalgebras from the set $\{A_1, ..., A_K\}$. If*

$$B_1 J B_2 J \cdots J B_t \neq 0 \tag{2}$$

then $\dim(B_1^{(0)} + \cdots + B_t^{(0)}) + \dim(B_1^{(1)} + \cdots + B_t^{(1)}) \leqslant p$.

Proof. If in the product (2) some subalgebra B_i appears two or more times then, since $JB_iJ \subseteq J$, we can reduce this product to get a non-zero product of the type (2) with the A_i's all distinct. ∎

Recall that $G(A) = G^{(0)} \otimes A^{(0)} + G^{(1)} \otimes A^{(1)}$ is the Grassmann envelope of A.

LEMMA 8. *Let $\lambda \geqslant h(l, d, t)$ where $l + d > p$ and $t > (l+d)m + q$. If f is a multilinear polynomial corresponding to a tableau T_λ, then $f \in Id(G(A))$.*

Proof. First notice that a non-zero multilinear polynomial f corresponding to T_λ generates in V_n and irreducible left FS_n-submodule. It follows that for any $r \in FS_n$ such that $r \circ f \neq 0$, $FS_n r \circ f = FS_n \circ f$. Therefore if we prove that for a suitable r, $0 \neq r \circ f \in Id(G(A))$ then $FS_n r \circ f \subseteq Id(G(A))$ and $f \in Id(G(A))$ will follow. The element r from the group algebra FS_n will be chosen during the proof.

Fix a basis, homogeneous in the $\mathbf{Z}_2$-grading, in each A_i, $i = 1, ..., K$, and also in J. Then the union of these bases $C = C^{(0)} \cup C^{(1)}$ is a homogenous basis for A. Now it is sufficient to show that $r \circ f$ evaluates to zero on all elements of the type $c \otimes g$, $c' \otimes g'$, $c \in C^{(0)}$, $c' \in C^{(1)}$, $g \in G^{(0)}$, $g' \in G^{(1)}$.

Let $c_1, ..., c_s$ be distinct basis elements from $C \cap B$ and $s > p$. Then any product containing all $c_1, ..., c_s$ and maybe some other elements from A is equal to zero as follows from Lemma 7. Hence, if we substitute

$c_1 \otimes g_1, ..., c_s \otimes g_s$ instead of some variables in $r \circ f$ then the resulting evaluation will be zero on $G(A)$. Therefore it is sufficient to prove that $r \circ f$ takes zero value only on elements of the type $c_1 \otimes g_1, ..., c_s \otimes g_s, e_1 \otimes h_1, e_2 \otimes h_2, ...$ where $c_1, ..., c_s$ are basic elements from $C \cap B$, $e_1, e_2 \cdots \in J$, $g_i, h_j \in G$, and all $c_1, ..., c_s$ belong to some semisimple subalgebra P of A with $\dim P^{(0)} + \dim P^{(1)} \leqslant p < l + d$.

Suppose first that $\dim P^{(0)} \leqslant d-1$. By Lemma 3 there exist $r \in FS_n$ and a subset of variables $Y = Y_1 \cup \cdots \cup Y_d$ such that $r \circ f \neq 0$ is symmetric on each set of variables Y_i for any i; there is also a decomposition $r \circ f = f_1 + f_2 + \cdots$ where $f_1, f_2, ...$ are polynomials alternating on suitable disjoint subsets of Y. Suppose that there exists some substitution in $r \circ f$ giving a non-zero value in $G(A)$. Then at least one of the summands f_i should have a non-zero evaluation. Let it be f_1. By Lemma 3 Y can be partitioned as $Y = Y'_1 \cup \cdots \cup Y'_{t+l}$ with $|Y_j| = d$ and f_1 is alternating on the variables of each subset Y'_j $j = 1, ..., t+l$. It follows that if for two variables y_1, y_2 from Y'_i we set $y_1 = c \otimes g_1$, $y_2 = c \otimes g_2$ where $c \in B^{(0)}$, $g_1, g_2 \in G^{(0)}$ then f_1 will be zero (being alternating on y_1 and y_2). On the other hand, since $\dim P^{(0)} \leqslant d-1$, in order to get a non-zero value for $r \circ f$ we need to substitute no more than $d-1$ elements of the type $c \otimes g$, $c \in B^{(0)}$, $g \in G^{(0)}$ in each set of variables Y'_i. This means that for a non-zero value we need to substitute in $r \circ f$ at least $t+l$ elements of the type $c \otimes g$, $c \in B^{(1)}$, $g \in G^{(1)}$ or $c \in J$, $g \in G$ instead of Y. But $J^q = 0$, hence, we should substitute no less than $l + t - q + 1 > dm$ elements of the type $c \otimes g$ with $c \in B^{(1)}$. It follows that for some $1 \leqslant i \leqslant d$ we will replace more than m variables in Y_i with elements $c \otimes g$ where c is a basis element from $B^{(1)}$. Since $m \geqslant \dim B^{(1)}$ it follows that there exist two variables $y_1, y_2 \in Y_i$ taking value $c \otimes g_1$ and $c \otimes g_2$ respectively, where $c \in B^{(1)}$, $g_1, g_2 \in G^{(1)}$. But $r \circ f$ is symmetric on y_1, y_2. Hence, the correspondence value will be zero.

Now assume that $\dim P^{(1)} \leqslant l-1$. As before by Lemma 4 there exist $r \in FS_n$ and a set of variables Y for $r \circ f$ such that $Y = Y_1 \cup \cdots \cup Y_l$, $r \circ f$ is alternating on each Y_i, $i = 1, ..., l$, and $r \circ f = f_1 + f_2 + \cdots$ where for any f_i there is a partition $Y = Y'_1 \cup \cdots \cup Y'_{t+d}$ on symmetric subsets of order l. If, for example, $f_1 \neq 0$ for some substitution then we should replace no less than $t + d - q + 1 > lm$ variables of Y with some $c \otimes g$, $c \in B^{(0)}$, $g \in G^{(0)}$ because f_1 is symmetric on $y_1, y_2 \in Y'_i$ and f_1 will be zero if $y_1 = c \otimes g_1$, $y_2 = c \otimes g_2$ with $c \in B^{(1)}$, $g_1, g_2 \in G^{(1)}$.

Since $\dim B^{(0)} \leqslant m$, we should replace some y_1, y_2 from the same alternating set Y_i to tensors $c \otimes g_1$, $c \otimes g_2$ respectively where $g_1, g_2 \in G^{(0)}$ and c is one of the basis elements of $B^{(0)}$. Hence $r \circ f$ will take a zero value and the proof is complete. ∎

Let $V_n \equiv FS_n = \bigoplus_{\lambda \vdash n} I_\lambda$ be the decomposition of the group algebra FS_n into its minimal two-sided ideals I_λ.

COROLLARY 1. *Let* $l+d=p+1$ *and* $t=2(m+1)m+1$ *where* $m=\dim A$. *Then* $\bigoplus_{\lambda \geqslant h(l,d,t)} I_\lambda \subseteq Id(G(A))$.

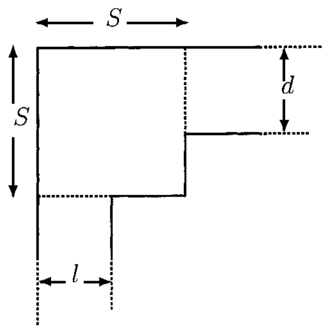
Proof. Let $\lambda \geqslant h(l,d,t)$. Since $2m+1 \geqslant p+1$ and $q \leqslant m$ where $J^q=0$ then $t>(2m+1)m+q \geqslant (p+1)m+q=(l+d)m+q$. Hence, by Lemma 8 for any tableau T_λ, $e_{T_\lambda} FS_n \subseteq Id(G(A))$. It follows that $I_\lambda = FS_n e_{T_\lambda} FS_n \subseteq Id(G(A))$. ∎

PROPOSITION 1. *Let* A *be a finite dimensional algebra and let* $p = \max(p^{(0)}+p^{(1)})$ *be defined as in* (1) *before Lemma* 7. *Then there exist constants* $C_1, r_1>0$ *depending only on* $\dim A$ *such that* $c_n(G(A)) \leqslant C_1 n^{r_1} p^n$.

Proof. Consider the decomposition of the nth cocharacter

$$\chi_n(G(A)) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda. \tag{3}$$

Suppose that $\lambda \vdash n$ is such that $\lambda \geqslant h(l,d,t)$ with $l+d=p+1$ and $t=2(m+1)m+1$ where $m=\dim A$. Then by Corollary 1, $m_\lambda=0$ for this λ. It follows that any diagram D_λ with non-zero m_λ in (3) lies in the union of some infinite hook $H(l,d)$ with $l+d=p$ and a square $S \times S$ where $S=2(m+1)m+1+m$ is a constant not depending on n (see picture).



In other words, D_λ contains a subdiagram $D_{\lambda'}$ such that $D_{\lambda'} \subset H(l,d)$, $l+d=p$, $\lambda \vdash n$, $\lambda' \vdash n'$ and $n-n' \leqslant T=S^2$. By Lemma 5 we have $d_\lambda \leqslant n^T d_{\lambda'}$. Denote by P the set of all $\lambda \vdash n$ such that $m_\lambda \neq 0$ in (3). By [1, Theorem 16] there exists a constant $k>0$ such that $m_\lambda \leqslant n^k$ for all $\lambda \in P$. Therefore, using Lemmas 5 and 6 we obtain

$$c_n(G(A)) = \sum_{\lambda \vdash n} m_\lambda d_\lambda \leqslant n^k \sum_{\substack{\lambda \vdash n \\ \lambda \in P}} d_\lambda$$

$$\leqslant n^k n^T \sum_{i=0}^{p} \sum_{n'=0}^{n} \sum_{\substack{\lambda' \vdash n' \\ \lambda' \in H(i,\, p-i)}} d'_\lambda$$

$$\leqslant n^{k+T} \sum_{i=0}^{p} \sum_{n'=1}^{n} C(n')^r\, p^{n'} \leqslant n^{k+T} C n^r p^n n(p+1).$$

So, $c_n(G(A)) \leqslant C_1 n^{r_1} p^n$ where $C_1 = (p+1)\, C$, $r_1 = k + T + r + 1$. ∎

5. IDENTITIES AND GRADED IDENTITIES

In this section we shall study the relations between the multilinear graded identities of A and of its Grassmann envelope $G(A)$.

The free algebra $F\langle X\rangle$ has a natural $\mathbf{Z}_2$-grading: we partition the set $X = Y \cup Z$ where $Y = \{y_1, y_2, ...\}$ and $Z = \{z_1, z_2, ...\}$ are countable disjoint sets; let $\mathscr{F}_0$ be the subspace of $F\langle X\rangle$ generated by all monomials on X of even degree in the variables of Z and $\mathscr{F}_1$ the subspace generated by all monomials of odd degree in the variables of Z. Then $(\mathscr{F}_0, \mathscr{F}_1)$ is the desired grading of $F\langle X\rangle$.

For a $\mathbf{Z}_2$-graded algebra A we denote with $Id^{gr}(A)$ the T_2-ideal of graded identities of A.

For $n_1, n_2 \geqslant 0$, let V_{n_1, n_2} be the space of multilinear polynomials in $y_1, ..., y_{n_1}, z_1, ..., z_{n_2}$. We define a linear isomorphism $\varphi\colon V_{n_1, n_2} \to V_{n_1, n_2}$ by the following rule: let $f \in V_{n_1, n_2}$ and write f as

$$f = \sum_{\substack{\sigma \in S_{n_2} \\ W}} \alpha_{\sigma,\, W} w_0 z_{\sigma(1)} w_1 \cdots w_{n_2 - 1} z_{\sigma(n_2)} w_{n_2}$$

where $W = (w_0, w_1, ..., w_{n_2})$ and all $w_0, w_1, ..., w_{n_2}$ are monomials in $y_1, ..., y_{n_1}$. Then

$$\varphi(f) = \tilde{f} = \sum_{\substack{\sigma \in S_{n_2} \\ W}} (-1)^\sigma \alpha_{\sigma,\, W} w_0 z_{\sigma(1)} w_1 \cdots w_{n_2 - 1} z_{\sigma(n_2)} w_{n_2}.$$

The symmetric groups S_{n_1} and S_{n_2} act independently on the left on the space V_{n_1, n_2}: S_{n_1} acts on $y_1, ..., y_{n_1}$ and S_{n_2} acts on $z_1, ..., z_{n_2}$. Let $R_1 = FS_{n_1}$, $R_2 = FS_{n_2}$ be the two corresponding group algebras. We shall compare the structure of V_{n_1, n_2} as a left module over R_1, R_2 and $R = R_1 \otimes R_2$. For $b = \sum_{\sigma \in S_{n_2}} \beta_\sigma \sigma$ we write $\tilde{b} = \sum_{\sigma \in S_{n_2}} (-1)^\sigma \beta_\sigma \sigma$.

LEMMA 9. *Let* $a \in R_1$, $b \in R_2$, $f \in V_{n_1, n_2}$. *Then*

1. $f \equiv 0$ *is a graded identity of* A *if and only if* $\tilde{f} \equiv 0$ *is a graded identity of* $G(A)$;
2. $\widetilde{bf} = \tilde{b}\tilde{f}$, $\widetilde{af} = a\tilde{f}$, $\tilde{\tilde{b}} = b$, $\tilde{\tilde{f}} = f$;
3. f *is alternating on some variables* $z_1, \ldots, z_m$ *if and only if* $\tilde{f}$ *is symmetric on* $z_1, \ldots, z_m$.

Proof. To prove the first part notice that if $f = f(y_1, \ldots, y_{n_1}, z_1, \ldots, z_{n_2})$, $u_1, \ldots, u_{n_1} \in A^{(0)}$, $v_1, \ldots, v_{n_2} \in A^{(1)}$, $g_1, \ldots, g_{n_1} \in G^{(0)}$, $h_1, \ldots, h_{n_2} \in G^{(1)}$ then

$$\begin{aligned}\tilde{f}(u_1 \otimes g_1, \ldots, u_{n_1} \otimes g_{n_1}, \ldots, v_1 \otimes h_1, \ldots, v_{n_2} \otimes h_{n_2}) \\ = f(u_1, \ldots, u_{n_1}, v_1, \ldots, v_{n_2}) \otimes g_1 \cdots g_{n_1} h_1 \cdots h_{n_2}.\end{aligned} \tag{4}$$

Now consider

$$f = \sum_{\substack{\sigma \in S_{n_2} \\ W}} \alpha_{\sigma, W} w_0 z_{\sigma(1)} w_1 \cdots w_{n_2 - 1} z_{\sigma(n_2)} w_{n_2}$$

and $b = \sum_{\tau \in S_{n_2}} \beta_\tau \tau$. Then

$$bf = \sum_{\substack{\sigma, \tau \in S_{n_2} \\ W}} \alpha_{\sigma, W} \beta_\tau w_0 z_{\tau\sigma(1)} w_1 \cdots w_{n_2 - 1} z_{\tau\sigma(n_2)} w_{n_2}$$

and

$$\begin{aligned}\widetilde{bf} &= \sum_{\substack{\sigma, \tau \in S_{n_2} \\ W}} (-1)^{\tau\sigma} \alpha_{\sigma, W} \beta_\tau w_0 z_{\tau\sigma(1)} w_1 \cdots w_{n_2 - 1} z_{\tau\sigma(n_2)} w_{n_2} \\ &= \left(\sum_{\tau \in S_{n_2}} (-1)^\tau \beta_\tau \tau \right) \\ &\quad \times \left(\sum_{\substack{\sigma \in S_{n_2} \\ W}} (-1)^\sigma \alpha_{\sigma, W} w_0 z_{\sigma(1)} w_1 \cdots w_{n_2 - 1} z_{\sigma(n_2)} w_{n_2} \right) = \tilde{b}\tilde{f}.\end{aligned}$$

All other statements in (2) and (3) are trivial. ∎

LEMMA 10. *Let* $B = B^{(0)} + B^{(1)}$ *be a* $\mathbf{Z}_2$*-graded algebra over* F *and* $d = \dim B^{(0)}$, $l = \dim B^{(1)}$. *Let* $f(y_1, \ldots, y_{dr}, z_1, \ldots, z_{ls}) \in V_{dr, ls}$ *be alternating in* r *disjoint subsets of variables* $\{y_1^i, \ldots, y_d^i\} \subseteq \{y_1, \ldots, y_{dr}\}$, $1 \leqslant i \leqslant r$, *and symmetric in* s *disjoint subsets of variables* $\{z_1^i, \ldots, z_l^i\} \subseteq \{z_1, \ldots, z_{ls}\}$, $1 \leqslant i \leqslant s$. *If* $f \notin Id^{gr}(G(B))$, *then there exist* $e_{T_\lambda} \in FS_{dr}$, $e_{T_\mu} \in FS_{ls}$, $\lambda = (r^d)$, $\mu = (l^s)$ *such that* $e_{T_\lambda} e_{T_\mu} f \notin Id^{gr}(G(B))$.

Proof. Let $n_1 = dr$, $n_2 = ls$ so that $f \in V_{n_1, n_2}$. Consider the left FS_{n_1}-module generated by f in V_{n_1, n_2} and its decomposition into the sum of irreducible FS_{n_1}-submodules. Since $f \notin Id^{gr}(G(B))$, there exists a Young tableau T_λ, $\lambda \vdash n_1$, such that $e_{T_\lambda} f \not\equiv 0$ on $G(B)$. If $l(\lambda) \geqslant r+1$ then $e_{T_\lambda} f$ is symmetric on at least $r+1$ variables among $y_1, ..., y_{dr}$. But all the y_i's are divided into r disjoint alternating subsets. Therefore $e_{T_\lambda} f = 0$ in $F\langle X \rangle$ being symmetric and alternating in two variables at the same time.

Assume now that $h(T_\lambda) \geqslant d+1$. In this case write $e_{T_\lambda} = e_1 e_2$ where $e_1 = \sum_{\sigma \in R_{T_\lambda}} \sigma$, $e_2 = \sum_{\tau \in C_{T_\lambda}} (-1)^\tau \tau$. Since the polynomial $e_2 f$ is alternating on some $d+1$ variables y_i's then also the polynomial $e_2 \tilde{f}$ is alternating on the same variables. Since $\dim B^{(0)} = d$ it follows that $\widetilde{e_2 \tilde{f}} \in Id^{gr}(B)$. Hence $e_{T_\lambda} \tilde{f} \equiv 0$ is also a graded identity of B. By Lemma 9, $\widetilde{e_{T_\lambda} \tilde{f}} = e_{T_\lambda} f \equiv 0$ is a graded identity of $G(B)$.

Since $n_1 = td$, we conclude that $e_{T_\lambda} f \notin Id^{gr}(G(B))$ only if D_λ is a rectangle with d rows and r columns, i.e., $\lambda = (r^d)$.

We now consider the left FS_{n_2}-submodule of V_{n_1, n_2} generated by f. As above there exists a Young tableau T_μ with $\mu \vdash n_2$ such that $e_{T_\mu} f \not\equiv 0$ on $G(B)$. Suppose first that $h(T_\mu) \geqslant s+1$ and write as before $e_{T_\mu} = e_1 e_2$ where $e_1 = \sum_{\sigma \in R_{T_\mu}} \sigma$, $e_2 = \sum_{\tau \in C_{T_\mu}} (-1)^\tau \tau$. In this case the action of e_2 on V_{n_1, n_2} is alternating on at least $s+1$ variables z_i's. But all the variables $\{z_1, ..., z_{ls}\}$ in f are divided into s symmetric disjoint subsets; hence, $e_2 f = 0$ in $F\langle X \rangle$ and $e_{T_\mu} f$ is also zero.

If, on the other hand, $l(T_\mu) \geqslant l+1$, then $g = e_{T_\mu} f$ is symmetric on $l+1$ variables z_i's. By Lemma 9 $\tilde{g}$ is alternating on the same $l+1$ odd variables; since $\dim B^{(1)} = l$, it follows that $\tilde{g} \in Id^{gr}(B)$. By using Lemma 9 again we then obtain that $e_{T_\mu} f = g = \tilde{\tilde{g}} \in Id^{gr}(G(B))$.

It follows that $e_{T_\mu} f \notin Id^{gr}(G(B))$ only if D_μ is a rectangle with s rows and l columns i.e., $\mu = (l^s)$.

We have proved that $f \notin Id^{gr}(G(B))$ implies $e_{T_\lambda} e_{T_\mu} f \notin Id^{gr}(G(B))$ where D_λ and D_μ are two rectangles of size $r \times d$ and $l \times s$ respectively. ∎

6. SIMPLE SUPERALGEBRAS AND THEIR GRASSMANN ENVELOPE

LEMMA 11. *Let $B = B^{(0)} + B^{(1)}$ be a simple $\mathbf{Z}_2$-graded algebra over an algebraically closed field F, $d = \dim B^{(0)}$, $l = \dim B^{(1)}$. Then for any positive integer t there exists λ such that $h(l, d, 2t - s) \leqslant \lambda \leqslant h(l, d, 2t)$, $s = \dim B$, and a tableau T_λ such that $G(B)$ does not satisfy an identify an identity $f \equiv 0$ corresponding to T_λ (i.e., $I_\lambda \not\subseteq Id(G(B))$).*

Proof. It is well known [7] that a simple finite dimensional associative $\mathbf{Z}_2$-graded algebra over an algebraically closed field of zero characteristic is isomorphic to one of the following algebras:

1. $M_{a,b}(F)=\left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{smallmatrix}\right)$ where A_{11}, A_{12}, A_{21}, A_{22} are $a\times a$, $a\times b$, $b\times a$ and $b\times b$ matrices respectively, $a>0$, $b\geqslant 0$, with grading

$$M_{a,b}^{(0)}(F)=\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22}\end{pmatrix}, \qquad M_{a,b}^{(1)}(F)=\begin{pmatrix} 0 & A_{12} \\ A_{21} & 0\end{pmatrix}.$$

2. $M_N(D)$, where $D=F+Fc$, $c^2=1$, with grading $(M_N(F), cM_N(F))$.

Suppose first that $B=M_{a,b}(F)$. Denote $d=\dim B^{(0)}$, $l=\dim B^{(1)}$. Then $\dim B=l+d$. Since B is the $(a+b)\times(a+b)$-matrix algebra over F, for any $t\geqslant 1$ there exists a multilinear polynomial $f=f(x_1^1, ..., x_{d+l}^1, ..., x_1^{2t}, ..., x_{d+l}^{2t})$ on $2t(d+l)$ variables such that f is alternating on the variables $x_1^i, ..., x_{d+l}^i$ for every $i=1, ..., 2t$ and $f\notin Id(B)$ (see [4]). Let E be a basis of B homogeneous in the $\mathbf{Z}_2$-grading. Then $|E|=d+l$ and, for every i, we need to substitute all elements from E instead of $x_1^i, ..., x_{d+l}^i$ in order to get a non-zero value for f. It means that after renumerating and renaming all the variables in f we can say that

$$f=f(y_1^1, ..., y_d^1, ..., y_1^{2t}, ..., y_d^{2t}, z_1^1, ..., z_l^1, ..., z_1^{2t}, ..., z_l^{2t})$$

is not a graded identity of B where y_i^j are even and z_i^j are odd variables. By Lemma 9 $\tilde{f}\equiv 0$ is not a graded identity of $G(B)$; moreover for every $i=1, ..., 2t$, since f is alternating in $y_1^i, ..., y_d^i$ and in $z_1^i, ..., z_l^i$, the polynomial $\tilde{f}$ is alternating in the variables $y_1^i, ..., y_d^i$ and symmetric in the variables $z_1^i, ..., z_l^i$.

Let $n_1=2td$, $n_2=2tl$. Then $\tilde{f}\in V_{n_1,n_2}$ and by the previous lemma it follows that there exist $e_{T_\lambda}\in R_1=FS_{n_1}$, $e_{T_\mu}\in R_2=FS_{n_2}$, $\lambda=((2t)^d)$, $\mu=(l^{2t})$ such that $g=e_{T_\lambda}e_{T_\mu}\tilde{f}\not\equiv 0$ on $G(B)$.

If $l=0$ then $g=e_{T_\lambda}\tilde{f}=e_{T_\lambda}f$ is the required non-identity since $\lambda=((2t)^d)=h(0, d, 2t)$. Therefore we may assume that $l>0$.

Let M be the $R_1\otimes R_2$-submodule of V_{n_1,n_2} generated by g; then M is isomorphic to the tensor product $M_1\otimes M_2$ where $M_1=R_1e_{T_\lambda}$, $M_2=R_2e_{T_\mu}$.

If we write $n=n_1+n_2$, then $V_{n_1,n_2}\subseteq V_n$ and we let $\bar{M}$ be the FS_n-submodule of V_n generated by M. Let

$$\bar{M}=\bar{M}_1\oplus\cdots\oplus\bar{M}_k$$

be its decomposition into FS_n-irreducibles. By the Littlewood–Richardson rule [6] every $\bar{M}_i$ is associated to a Young diagram D_λ such that $h(l, d, 2t-s)\leqslant\lambda\leqslant h(l, d, 2t)$ where $s=\max\{l, d\}$. Therefore $\lambda\geqslant h(l, d, 2t-s)$

for $s = \dim B$. Since $\bar{M}$ is not contained in the T-ideal of ordinary (non-graded) identities of $G(B)$, it follows that for some multilinear $u \in \bar{M}$ we must have $e_{T_\lambda} u \not\equiv 0$ on $G(B)$ and the proof of the lemma is completed in case $B = M_{a,b}(F)$.

Now assume $B = M_N(D)$, $B^{(0)} = M_N(F)$, $B^{(1)} = cM_N(F)$ with $c^2 = 1$. Then $l = d = N^2 = \dim B$. As in the previous case we let $f_0 = f_0(x_1^1, ..., x_d^1, ..., x_1^{2t}, ..., x_d^{2t})$ be a multilinear polynomial which is alternating on the variables $x_1^i, ..., x_d^i$, $i = 1, ..., 2t$ and $f_0 \notin Id(B)$. If we set

$$f = f_0(y_1^1, ..., y_d^1, ...y_1^{2t}, ..., y_d^{2t})\, f_0(z_1^1, ..., z_l^1, ..., z_1^{2t}, ..., z_l^{2t}).$$

Then it is clear that $f \equiv 0$ is not a graded identity of B. The same arguments as in the previous case complete the proof of the lemma. ∎

LEMMA 12. *Let $B = B^{(0)} + B^{(1)}$ be a finite dimensional simple $\mathbf{Z}_2$-graded algebra over an algebraically closed field F. Then for any non-zero homogeneous elements $\bar{b} \in B$ and for any matrix unit E_{ij} there exist homogeneous elements $a, c \in B$ such that $a\bar{b}c = E_{ij}$.*

Proof. Obviously, up to scalars one can take $a = E_{i\alpha}$, $c = E_{\beta j}$ for suitable α, β. ∎

We recall the notation: if A is a finite dimensional $\mathbf{Z}_2$-graded algebra over F, then $A = B + J$, $B = A_1 \oplus \cdots \oplus A_K$ where $A_1, ..., A_K$ are simple $\mathbf{Z}_2$-graded subalgebras of A and J is the Jacobson radical. As before, we consider a non-zero product

$$B_1 J B_2 J \cdots J B_k \neq 0 \tag{5}$$

where $B_1, ..., B_k$ are distinct subalgebras from the set $\{A_1, ..., A_K\}$.

LEMMA 13. *Let A be a finite dimensional $\mathbf{Z}_2$-graded algebra over an algebraically closed field F and suppose that $B_1 J B_2 J \cdots J B_k \neq 0$. Let $f_1, ..., f_k$ be multilinear polynomials on distinct sets of variables such that for every $i = 1, ..., k$, $f_i \notin Id(G(B_i))$. Then the multilinear polynomial*

$$u_1 f_1 v_1 w_1 u_2 f_2 v_2 w_2 \cdots w_{k-1} u_k f_k v_k \tag{6}$$

where $u_1, v_1, w_1, ..., w_{k-1}, u_k, v_k$ are new variables, is not an identity of $G(A)$.

Proof. By (5) there exist matrix units $b_1 \in B_1, ..., b_k \in B_k$, $e_1, ..., e_{k-1} \in J$ (all $b_1, ..., b_k$, $e_1, ..., e_{k-1}$ are homogeneous in the $\mathbf{Z}_2$-grading) such that

$$b_1 e_1 b_2 e_2 \cdots e_{k-1} b_k \neq 0 \tag{7}$$

in A. Let $f_i = f_i(x^i_1, ..., x^i_{n_i})$. Since f_i is not an identity of $G(B_i)$, there exist homogenous $\bar{x}^i_1, ..., \bar{x}^i_{n_i} \in B_i$, $g^j_i \in G$ such that $f_i(\bar{x}^i_1 \otimes g^i_1, ..., \bar{x}^i_{n_i} \otimes g^i_{n_i}) \neq 0$. We shall say that x^i_j is an even variable if $\bar{x}^i_j \in B^{(0)}_i$ and x^i_j is an odd variable is case $\bar{x}^i_j \in B^{(1)}_i$. So, we can regard f_i as a graded polynomial and

$$f_i(\bar{x}^i_1 \otimes g^i_1, ..., \bar{x}^i_{n_i} \otimes g^i_{n_i}) = \tilde{f}_i(\bar{x}^i_1, ..., \bar{x}^i_{n_i}) \otimes g^i_1 \cdots g^i_{n_i}.$$

It is obvious that $\tilde{f}_i(\bar{x}^i_1, ..., \bar{x}^i_{n_i}) = \bar{b}_i \neq 0$, $\bar{b}_i \in B_i$. By Lemma 12 one can choose homogeneous elements $a_i, c_i \in B_i$ such that $a_i \bar{b}_i c_i = b_i$. Therefore the polynomial $u_i \tilde{f}_i v_i$ takes the value b_i by evaluating $u_i, x^i_1, ..., x^i_{n_i}, v_i$ in a_i, $\bar{x}^i_1, ..., \bar{x}^i_{n_i}$, c_i respectively. Let h_i, h'_i, t_i be elements of G of the same homogeneous degree as a_i, e_i, c_i respectively. Then for $i = 1, ..., k-1$, we get

$$\begin{aligned}&(a_i \otimes h_i)\, f_i(\bar{x}^i_1 \otimes g^i_1, ..., \bar{x}^i_{n_i} \otimes g^i_{n_i})(c_i \otimes h'_i)(e_i \otimes t_i)\\ &\quad = a_i \tilde{f}_i(\bar{x}^i_1, ..., \bar{x}^i_{n_i})\, c_i e_i \otimes h_i g^i_1 \cdots g^i_{n_i} h'_i t_i = b_i e_i \otimes h_i g^i_1 \cdots g^i_{n_i} h'_i t_i.\end{aligned}$$

For $i = k$ similarly we get

$$(a_k \otimes h_k)\, f_i(\bar{x}^k_1 \otimes g^k_1, ..., \bar{x}^k_{n_k} \otimes g^k_{n_k})(c_k \otimes h'_k) = b_k \otimes h_k g^k_1 \cdots g^k_{n_k} h'_k.$$

Since G is the infinite dimensional Grassmann algebra, we can choose homogeneous elements h_i, h'_i, t_i, g^j_i in G such that

$$h_1 g^1_1 \cdots g^1_{n_1} h'_1 t_1 h_2 g^2_1 \cdots g^2_{n_2} h'_2 t_2 \cdots t_{k-1} h_k g^k_1 \cdots g^k_{n_k} h'_k \neq 0. \tag{8}$$

Hence, by (7) and (8) the polynomial $u_1 f_1 v_1 w_1 u_2 f_2 v_2 w_2 \cdots w_{k-1} u_k f_k v_k$ takes non-zero value on $u_i = a_i \otimes h_i$, $v_i = c_i \otimes h'_i$, $w_i = e_i \otimes t_i$, $x^j_i = \bar{x}^j_i \otimes g^j_i$, and the proof of the lemma is complete. ∎

7. GLUING YOUNG TABLEAUX

Let $\lambda_1 \vdash n_1$, $\lambda_2 \vdash n_2, ..., \lambda_k \vdash n_k$ be given partitions and suppose that they satisfy the following conditions

$$h(l_i, d_i, t_i - s_i) \leqslant \lambda_i \leqslant h(l_i, d_i, t_i), \qquad i = 1, ..., k \tag{9}$$

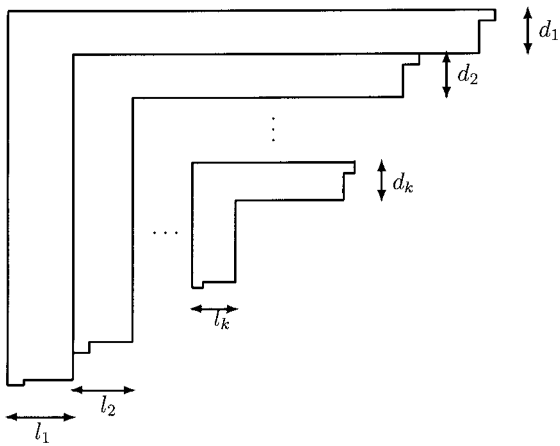
and

$$t_i - s_i \geqslant t_{i+1} + l_{i+1}, t_{i+1} + d_{i+1}, \qquad i = 1, ..., k-1. \tag{10}$$

Let $D_1 = D_{\lambda_1}$, $D_2 = D_{\lambda_2}, ..., D_k = D_{\lambda_k}$ be the corresponding Young diagrams. By (9) the length of the first d_i rows of D_i is no less than $l_i + t_i - s_i$ and no more than $l_i + t_i$. Similarly, the length of the first l_i

columns (if $l_i > 0$) is no less than $d_i + t_i - s_i$ and no more than $d_i + t_i$. The inequalities (10) mean that if we glue the 1st row of D_{i+1} to the $(d_i + 1)$th row of D_i, the 2nd row of D_{i+1} to the $(d_i + 2)$th row of D_i and so on, then we will get as a result a new Young diagram $D_i \star D_{i+1}$ with $n_i + n_{i+1}$ boxes.

Consider the diagram $D_\lambda = D_1 \star D_2 \star \cdots \star D_k$ obtained by "gluing" together $D_1, D_2, ..., D_k$ as above (see picture).



Obviously, $\lambda \leqslant h(l, d, t)$ where $l = l_1 + \cdots + l_k$, $d = d_1 + \cdots + d_k$ and $t \geqslant t_1 + l_1 - l$, $t_1 + d_1 - d$. On the other hand $\lambda \geqslant h(l, d, t_k - s_k)$.

Let now $T_1, ..., T_k$ be Young tableaux corresponding to $\lambda_1, ..., \lambda_k$ respectively. If $\lambda_1, ..., \lambda_k$ satisfy (9) and (10) above we can glue the tableaux in a similar way: if α_{uv} is the entry appearing in the (u, v) position of T_i, we write $T_i = D_i(\alpha_{uv})$. For every $i = 2, ..., k$ we now add $n_1 + \cdots + n_{i-1}$ to all the entries of T_i, obtaining in this way a new tableau $D_i(\alpha_{uv} + n_1 + \cdots + n_{i-1})$. If $T_1 = D_1(\alpha_{uv})$, $T_2 = D_2(\beta_{uv})$, ..., $T_k = D_k(\gamma_{uv})$, we then define

$$\begin{aligned} T_\lambda &= T_1 \star T_2 \star \cdots \star T_k \\ &= D_1(\alpha_{uv}) \star D_2(\beta_{uv} + n_1) \star \cdots \star D_k(\gamma_{uv} + n_1 + \cdots + n_{k-1}). \end{aligned}$$

It is clear that the tableau so obtained is in the distinct entries $1, 2, ..., n$ where $n = n_1 + \cdots + n_k$.

Define $N_1=\{1, ..., n_1\}$ and, for $2\leqslant i\leqslant k$, $N_i=\{n_1+\cdots+n_{i-1}+1, ..., n_1+\cdots+n_i\}$. Thus $N=\{1, ..., n\}$ is the disjoint union $N=N_1\cup\cdots\cup N_k$. For $i=1, ..., k$, we think of S_{n_i} as the permutation group acting on the set N_i, so that we can consider the group algebras $FS_{n_1}, ..., FS_{n_k}$ as embedded in FS_n with one-dimensional intersection.

We need to relate the essential idempotent e_{T_λ} to $e_{T_1}, ..., e_{T_k}$; we do so in the next lemma.

Lemma 14. *Suppose that $\lambda_1, ..., \lambda_k$ satisfy the conditions* (9) *and* (10) *and let $T_1, ..., T_k$ be corresponding tableaux. If $T_\lambda=T_1\bigstar\cdots\bigstar T_k$, then*

$$e_{T_\lambda}=e_{T_1}\cdots e_{T_k}+b$$

where b is a linear combination of $\sigma\in S_n$ such that $\sigma(N_i)\nsubseteq N_i$ for some $1\leqslant i\leqslant k$.

Proof. Let $E=\{\sigma\in S_n\mid\sigma(N_i)\subseteq N_i$ for all $i=1, ..., k\}$. Obviously, $E=S_{n_1}\times\cdots\times S_{n_k}$ in our notation. We need to check that

$$e_{T_\lambda}-e_{T_1}\cdots e_{T_k}=\sum_{\sigma\in S_n\setminus E}\alpha_\sigma\sigma$$

for suitable $\alpha_\sigma\in F$. Recall that

$$e_{T_\lambda}=\sum_{\substack{\sigma\in R_{T_\lambda}\\ \tau\in C_{T_\lambda}}}(-1)^\tau\,\sigma\tau \tag{11}$$

where R_{T_λ} is the subgroup of S_n of row permutations of T_λ and C_{T_λ} is the subgroup of column permutations of T_λ. Denote $R=R_{T_\lambda}\cap E$, $C=C_{T_\lambda}\cap E$ and

$$R_i=\{\sigma\in R\mid\sigma(x)=x\ \forall x\in N\setminus N_i\},$$

$$C_i=\{\sigma\in C\mid\sigma(x)=x\ \forall x\in N\setminus N_i\},$$

One can split the sum (11) into two parts, $e_{T_\lambda}=u+w$, where

$$u=\sum_{\substack{\sigma\in R\\ \tau\in C}}(-1)^\tau\,\sigma\tau=\left(\sum_{\sigma\in R}\sigma\right)\left(\sum_{\tau\in C}(-1)^\tau\,\tau\right)$$

and w contains all the remaining terms in the right hand side of (11). We will show that $u=e_{T_1}\cdots e_{T_k}$ and w is a linear combination of $\sigma\notin E$.

First note that any $\sigma\in R$ has a decomposition $\sigma=\sigma_1\cdots\sigma_k$ where $\sigma_i\in R_i$, $i=1, ..., k$. Moreover, $\sigma=\sigma'=\sigma_1'\cdots\sigma_k'$ if and only if $\sigma_1=\sigma_1', ..., \sigma_k=\sigma_k'$.

On the other hand, if $\sigma_1, ..., \sigma_k$ are some permutations from $R_1, ..., R_k$ respectively then $\sigma = \sigma_1 \cdots \sigma_k$ belongs to R. Hence,

$$\sum_{\sigma \in R} \sigma = \sum_{\sigma_1 \in R_1, ..., \sigma_k \in R_k} \sigma_1 \cdots \sigma_k = \left(\sum_{\sigma_1 \in R_1} \sigma_1 \right) \cdots \left(\sum_{\sigma_k \in R_k} \sigma_k \right). \tag{12}$$

Similarly,

$$\begin{aligned} \sum_{\tau \in C} (-1)^{\tau} \tau &= \sum_{\tau_1 \in C_1, ..., \tau_k \in C_k} (-1)^{\tau_1 \cdots \tau_k} \tau_1 \cdots \tau_k \\ &= \left(\sum_{\tau_1 \in C_1} (-1)^{\tau_1} \tau_1 \right) \cdots \left(\sum_{\tau_k \in C_k} (-1)^{\tau_k} \tau_k \right). \end{aligned} \tag{13}$$

Since for $i \neq j$, FS_{n_i} and FS_{n_j} commute elementwise in FS_n and

$$\left(\sum_{\sigma \in R_i} \sigma \right) \left(\sum_{\tau \in C_i} (-1)^{\tau} \tau \right) = e_{T_i}$$

we obtain from (12) and (13) that $u = e_{T_1} \cdots e_{T_k}$.

Consider now elements $\tau \in C_{T_\lambda} \backslash E$ and $\sigma \in R_{T_\lambda}$. Then for this τ there exists i such that $\tau(x) \in N_j$ for some $x \in N_i$ and $j < i$. Suppose that x belongs to the pth row of T_λ. Then $\tau(x)$ lies in a higher row (say in the qth row) of T_λ since $\tau(x) \in N_j$ and $j < i$. By the construction of T_λ all entries of the qth row belong to $N_1 \cup \cdots \cup N_j$. Hence, $\sigma\tau(x) \notin N_i$ for all $\sigma \in R_{T_\lambda}$ and $\sigma\tau \notin E$. On the other hand, if $\tau \in C_{T_\lambda} \cap E$, $\sigma \in R_{T_\lambda} \backslash E$ then $\sigma\tau \notin E$ since E is a subgroup of S_n and $\tau \in E$. It follows that $w = \sum_{\sigma \notin E} \alpha_\sigma \sigma$ and the proof of the lemma is complete. ▮

8. COMPUTING THE LOWER BOUND

As before we assume that whenever the group S_n acts on a multilinear polynomial on $m \geqslant n$ variables, it acts only on the first n variables.

LEMMA 15. *Let A be a finite dimensional $\mathbf{Z}_2$-graded algebra over an algebraically closed field F with Jacobson radical J. Let $B_1, ..., B_k$ be distinct $\mathbf{Z}_2$-graded simple subalgebras of A such that $B_1 J B_2 J \cdots J B_k \neq 0$ and let $d = \dim B_1^{(0)} \oplus \cdots \oplus B_k^{(0)}$, $l = \dim B_1^{(1)} \oplus \cdots \oplus B_k^{(1)}$. Then for any positive integer $t \geqslant 2 \dim A$ there exists $\lambda \vdash n$ such that $h(l, d, 2t - s) \leqslant \lambda \leqslant h(l, d, 2t)$, $s = 4 \dim A$, and for some tableau T_λ, $e_{T_\lambda} \circ f \notin Id(G(A))$ for some multilinear polynomial f with $\deg f \leqslant n + 3 \dim A$.*

Proof. Denote $d_i = \dim B_i^{(0)}$, $l_i = \dim B_i^{(1)}$, $i = 1, ..., k$. Then $d = d_1 + \cdots + d_k$, $l = l_1 + \cdots + l_k$. By Lemma 11 for any integer t_i there exists a partition λ_i such that $h(l_i, d_i, 2t_i - s_i) \leqslant \lambda_i \leqslant h(l_i, d_i, 2t_i)$ where $l_i, d_i \leqslant s_i = \dim B_i$ and a tableau T_i on λ_i such that $g_i \notin Id(G(B_i))$ for some multilinear polynomial g_i corresponding to T_i. We choose $t_1, ..., t_k$ by the following rule. Let $t_1 = t \geqslant 2 \dim A$ be arbitrary. Denote $q_i = s_{i-1} + \max\{l_i, d_i\}$, $i = 2, ..., k$, and set $q'_i = q_i$ if q_i is even and $q'_i = q_i + 1$ if q_i is odd. Then define $2t_{i+1} = 2t_i - q'_{i+1}$, $i = 1, ..., k-1$. It follows that $2t_i - s_i = 2t_{i+1} + q'_{i+1} - s_i \geqslant 2t_{i+1} + \max\{l_{i+1}, d_{i+1}\}$. Hence $\lambda_1, ..., \lambda_k$ satisfy (9) and (10) with $t_1, ..., t_k$ replaced by $2t_1, ..., 2t_k$.

We now glue the above tableaux T_1 on $\lambda_1, ..., T_k$ on λ_k as shown in the previous section. We obtain a partition λ and a tableau $T_\lambda = T_1 \star \cdots \star T_k$. Moreover $h(l, d, 2t_k - s_k) \leqslant \lambda \leqslant h(l, d, u)$ for every $u \geqslant 2t_1 + l_1 - l$, $2t_1 + d_1 - d$. We now compute

$$\begin{aligned} 2t_1 - 2t_k &= \sum_{i=1}^{k-1} (2t_i - 2t_{i+1}) = \sum_{i=1}^{k-1} q'_{i+1} \\ &\leqslant k + \sum_{i=1}^{k-1} q_{i+1} \leqslant k + \sum_{i=1}^{k-1} (s_i + s_{i+1}) \\ &\leqslant k + 2\dim(B_1 \oplus \cdots \oplus B_k) \leqslant 3 \dim A \end{aligned} \tag{14}$$

Hence $2t_k - s_k \geqslant 2t - 3\dim A - s_k \geqslant 2t - 4\dim A$ and, so, λ satisfies the inclusions

$$h(l, d, 2t - 4\dim A) \leqslant \lambda \leqslant h(l, d, 2t).$$

For every $i = 1, ..., k$, let T_i be the tableau on λ_i such that $g_i \notin Id(G(B_i))$ for some multilinear polynomial corresponding to T_i (see Lemma 11) and let $n_i = \deg g_i$. Write $n = n_1 + \cdots + n_k$ and let $\{1, ..., n\} = N_1 \cup \cdots \cup N_k$ where the N_i's are defined as in the previous section. For every $i = 1, ..., k$, we denote by f_i the multilinear polynomial g_i written in the variables x_j where $j \in N_i$.

We now construct the multilinear polynomial

$$f = u_1 f_1 v_1 w_1 u_2 f_2 v_2 w_2 \cdots w_{k-1} u_k f_k v_k$$

where the u_i, v_i, w_i are new variables. By Lemma 13 $f \notin Id(G(A))$; therefore f is non-zero under some substitution $\theta(w_i) = \bar{w}_i$, $\theta(u_i) = \bar{u}_i$, $\theta(v_i) = \bar{v}_i$, $\theta(x_i) = \bar{x}_i$ where $\bar{w}_1, ..., \bar{w}_{k-1} \in J \otimes G$, $\bar{u}_i, \bar{v}_i \in B_i \otimes G$ and $\bar{x}_j \in B_i \otimes G$ if $j \in N_i$. So, $\theta(f) \neq 0$. Notice that $\deg f = n + 3k - 1 \leqslant n + 3\dim A$.

To complete the proof of the lemma we next show that $e_{T_\lambda} \circ f \notin Id(G(A))$. Indeed, consider the same substitution θ as above. Then by Lemma 14

$$\theta(e_{T_\lambda} \circ f) = \theta(e_{T_1} \cdots e_{T_k} \circ f) + \theta(b \circ f)$$

where b is a linear combination of $\sigma \in S_n$ such that σ "shuffles" the sets $N_1, ..., N_k$. Since e_{T_i} is an essential idempotent i.e., $e_{T_i}^2 = \mu_i e_{T_i}$ for some $\mu \in \mathbf{Z}$, $\mu_i \neq 0$, and the multilinear polynomial f_i corresponds to T_i then we have that $e_{T_i} \circ f_i = \mu_i f_i$. Hence $\theta(e_{T_1} \cdots e_{T_k} \circ f) = \mu_1 \cdots \mu_k \theta(f) \neq 0$.

On the other hand we now prove that every summand in $\theta(b \circ f)$ is equal to zero in $G(A)$. In fact, let $\sigma \in S_n$ be such that $\sigma(N_i) \not\subset N_i$ for some $1 \leqslant i \leqslant k$. Then $\sigma(j) \in N_q$ for some $j \in N_i$, $q \neq i$; this says that $\theta(\sigma \circ f_i)$ belongs to $G(B_q)$. We have

$$\theta(\sigma \circ f) = \bar{u}_1 \theta(\sigma \circ f_1)\, \bar{v}_1 \bar{w}_1 \cdots \theta(\sigma \circ f_k)\, \bar{v}_k = 0$$

since $\bar{u}_i \theta(\sigma \circ f_i) \in G(B_i)\, G(B_q) \subseteq B_i B_q \otimes G = 0$. Therefore $\theta(b \circ f) = 0$ on $G(A)$ and the proof is complete. ∎

As a corollary of the previous result we get

Proposition 2. *Let A be a finite dimensional algebra over an algebraically closed field of characteristic zero. Let also $p = \max(p^{(0)} + p^{(1)})$ be defined as in* (1) *before Lemma 7. Then there exist constants C_2, $r_2 > 0$ depending only on* $\dim A$ *such that $c_n(G(A)) \geqslant C_2 n^{-r_2} p^n$.*

Proof. Let $p = p^{(0)} + p^{(1)}$ and write $p^{(0)} = d$, $p^{(1)} = l$. Let also $m = \dim A$. For any $N > 5m^2 + 3m$ divide $N - dl - 3m$ by $2p$: we get $N = 2tp + dl + 3m + r$ for some $t > 2m$ and $0 \leqslant r < 2p$.

By Lemma 15 there exists n, $2tp - 4mp + dl \leqslant n \leqslant 2tp + dl$ and $\lambda \vdash n$, $h(l, d, 2t - 4m) \leqslant \lambda \leqslant h(l, d, 2t)$ such that $e_{T_\lambda} \circ f \notin Id(G(A))$ for some tableau T_λ and multilinear polynomial f with $n \leqslant \deg f = c \leqslant n + 3m$.

Write $N = c + u$ and construct the polynomial $f' = f \cdot x_{c+1} \cdots x_N$, where $x_{c+1}, ..., x_N$ are new variables distinct from the ones appearing in f. By Lemma 13 and Lemma 15 (or their proof), it is easy to see that $e_{T_\lambda} \circ f' \notin Id(G(A))$. By the branching rule of the symmetric group, it follows that

$$FS_N e_{T_\lambda} \circ f' \subseteq \bigoplus_{\substack{\mu \vdash N \\ \mu \geqslant \lambda}} I_\mu \circ f'$$

where I_μ is the two-sided ideal of FS_N corresponding to μ. Hence there exists $\mu \geqslant \lambda$ and a tableau T_μ such that $FS_N e_{T_\mu} \circ f' \nsubseteq Id(G(A))$.

We have: $c_N(G(A)) \geqslant d_\mu \geqslant d_{h(l,\,d,\,2t-4m)}$. Since $N - |h(l, d, 2t-4m)| = N - (2tp - 4mp + dl) < 8m^2 + 7m$ is bounded and, by [2, 7.16], $d_{h(l,\,d,\,2t-4m)} \underset{n\to\infty}{\simeq} an^b(l+d)^n$ it follows that $c_N(G(A)) \geqslant CN^u p^N$ for some constants C, u depending on m. ∎

9. THE MAIN RESULT

We can now prove our main theorem

THEOREM 3. *Let A be a PI-algebra over any field F of characteristic zero. Then $Inv(A)$ exists and is an integer.*

Proof. If K is an extension field of F then it is not difficult to prove that $c_n(A) = c_n(A \otimes_F K)$ (see [5, Remark 1]). Therefore we may assume that F is algebraically closed. By Kemer's theorem (Theorem 2 above) there exists a finite dimensional algebra B over F such that $Id(A) = Id(G(B))$; hence the conclusion now follows from Proposition 1 and Proposition 2 since then $C_2 n^{r_2} p^n \leqslant c_n(G(B)) \leqslant C_1 n^{r_1} p^n$, for some constants C_1, C_2, r_1, r_2 and $Inv(A) = p$. ∎

Kemer introduced the verbally prime T-ideals as basic blocks for the study of arbitrary T-ideals (see [7]). An algebra A is said verbally prime if $Id(A)$ is verbally prime. It turns out that the verbally prime algebras are: F, $F\langle X\rangle$, $M_k(F)$, $M_k(G)$ and $M_{k,l}(G)$ $(0 < l \leqslant [k/2])$ where

$$M_{k,l}(G) = \begin{array}{c c} & \begin{array}{cc} k & l \end{array} \\ \begin{array}{c} k \\ l \end{array} & \begin{pmatrix} G^{(0)} & G^{(1)} \\ G^{(1)} & G^{(0)} \end{pmatrix} \end{array}.$$

COROLLARY 2 [3, 9]. 1. $Inv(M_k(F)) = k^2$;

2. $Inv(M_k(G)) = 2k^2$;

3. $Inv(M_{k,l}(G)) = (k+l)^2$.

Proof. To prove (2), notice that $M_k(G) \simeq M_k(F) \otimes G^{(0)} + M_k(F) \otimes G^{(1)}$ is the Grassmann envelope of $M_k(F)$ with grading $(M_k(F), M_k(F))$. Hence $Inv(M_k(G)) = \dim M_k(F) + \dim M_k(F) = 2k^2$.

We prove (3) assuming that $l \geqslant 0$ and this will also prove (1). Now, $M_{k,l}(G) \simeq M_{k,l}^{(0)}(F) \otimes G^{(0)} + M_{k,l}^{(1)}(F) \otimes G^{(1)}$ is the Grassmann envelope of $M_{k,l}(F)$ with grading $(M_{k,l}^{(0)}(F), M_{k,l}^{(1)}(F))$. Hence $Inv(M_{k,l}(G)) = \dim M_{k,l}^{(0)}(F) + \dim M_{k,l}^{(1)}(F) = k^2 + l^2 + 2kl = (k+l)^2$. ∎

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